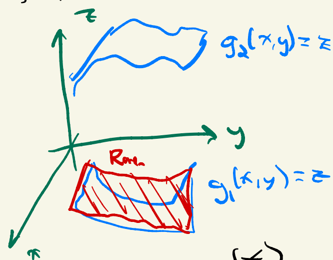
Last time: triple integrals + center of mass + 1st + 2nd moments



Mass $M = \iiint_R \rho(x,y,z)\, dV$

↑ density

Center of mass

"weighted average position":

$\mathbb{R}^1$

$$\bar{x} = \frac{1}{M}\int_{[a,b]} x\,\rho(x)\, dx$$

density [mass/length]

$\mathbb{R}^2$

$$M_y = \bar{x} = \frac{1}{M}\iint_R x\,\sigma(x,y)\, dA$$

$$M_x = \bar{y} = \frac{1}{M}\iint_R y\,\sigma(x,y)\, dA$$

"1st moment"

$\mathbb{R}^3$

$$M_{yz} = \bar{x} = \frac{1}{M}\iiint_R x\,\rho(x,y,z)\, dV$$

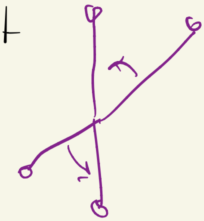
$$M_{xz} = \bar{y} = \frac{1}{M}\iiint_R y\,\rho(x,y,z)\, dV$$

$$M_{xy} = \bar{z} = \frac{1}{M}\iiint_R z\,\rho(x,y,z)\, dV$$

moments of planes

2nd moment application:

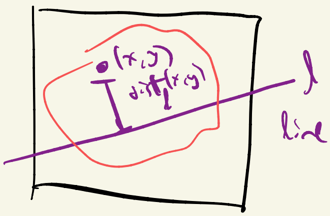
Linear moment		Angular moment
· Force	$\Longleftrightarrow$	Torque
· Momentum	$\Longleftrightarrow$	angular momentum
· Mass	$\Longleftrightarrow$	Moment of inertia
· Acceleration	$\Longleftrightarrow$	angular acceleration



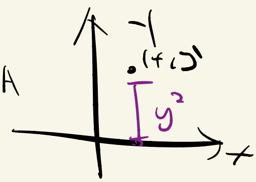
· 2nd moment: pick axis of rotation. Then just weight by distance^2!

$\mathbb{R}^2$:

$$I_\ell = \iint_R (\text{dist}_\ell(x,y))^2 \sigma(x,y)\, dA$$

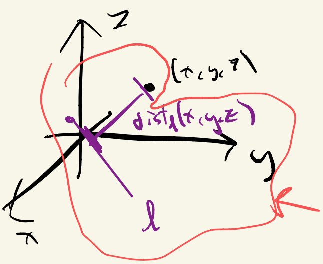


(ex: $I_x = \iint_R y^2 \sigma(x,y)\, dA$



$\mathbb{R}^3$:

$$I_\ell = \iiint_R (\text{dist}_\ell(x,y,z))^2 \rho(x,y,z)\, dV$$

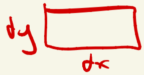


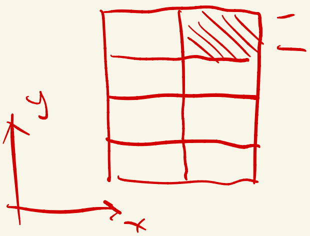
Think about flipping and angular momentum!

Will save problems for E.C. Tuesday!

15.7 Cylindrical coords

- In rectangular:

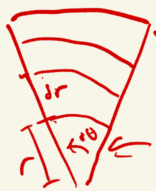




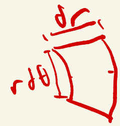
$dA = dx\,dy$ (Area of rectangle)

- In polar

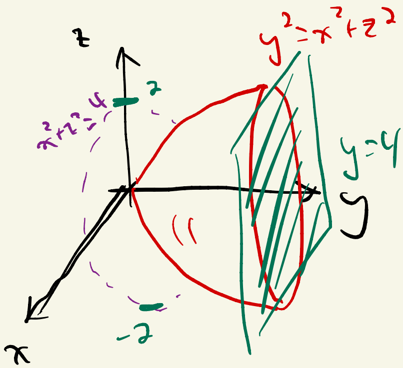
$dA = r\,dr\,d\theta$



different areas (farther from origin = bigger area)



Recall last problem in prev lecture:

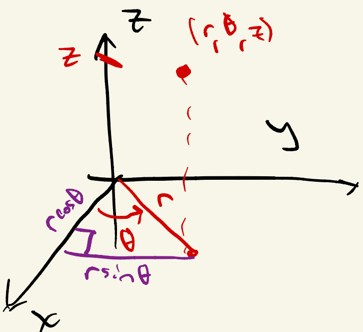


$$\int_{z=-2}^{z=2} \int_{x=-\sqrt{4-z^2}}^{x=\sqrt{4-z^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2+y^2}\; dy\, dx\, dz$$

polar in xz (integrating circle)

rectangular

Cylindrical: do polar in xy plane (or xz or yz)
rectangular in z axis (or y or x)



$r \geq 0$

$0 \leq \theta \leq 2\pi$

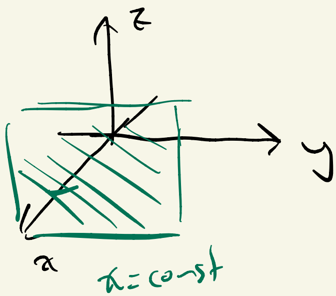
$z = z$

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\}\text{ cylindrical to rectangular}$$

$$\left.\begin{aligned} r^2 &= x^2 + y^2 \\ \theta &= \arctan\left(\frac{y}{x}\right) \end{aligned}\right\}\text{ rectangular to cylindrical}$$

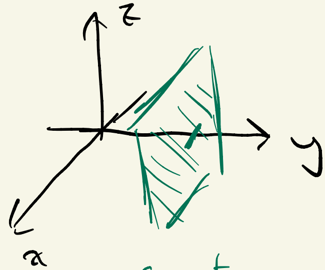
Constant coordinate surfaces

· In rectangular:

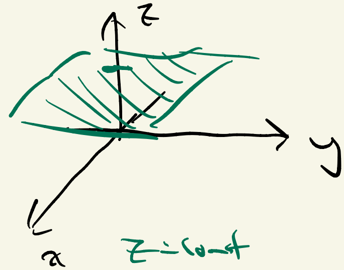


$x = \text{const}$

planes!

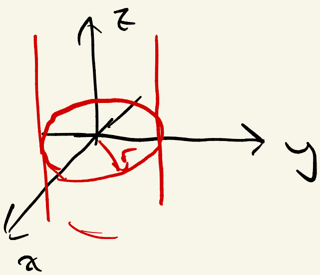


$y = \text{const}$



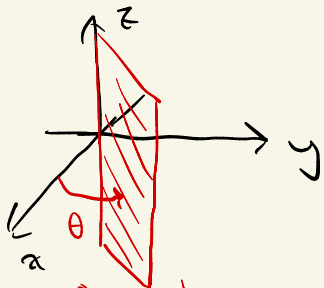
$z = \text{const}$

Cylindrical



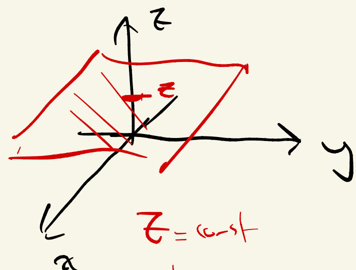
$r = \text{const}$

infinite cylinders



$\theta = \text{const}$

half plane



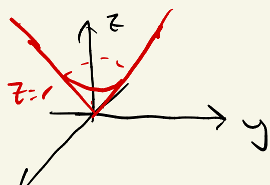
$z = \text{const}$

plane

· Other useful surfaces for easy integration (rectangular in green)

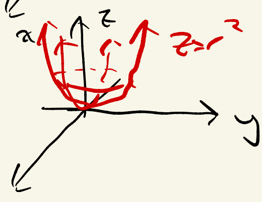
· $z = r$ (infinite) cone

$\left(z = \sqrt{x^2+y^2} \right)$



· $z = r^2$ paraboloids

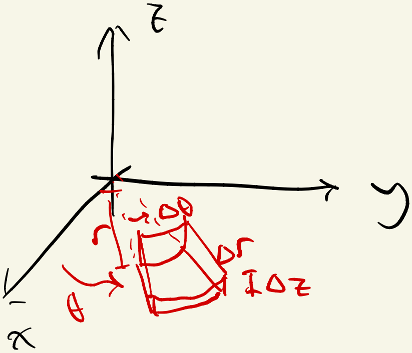
$\left(z = x^2+y^2 \right)$



Riemann Sums

$$\iiint f\,dV \simeq \sum_{\substack{\text{tiny} \\ \text{volumes}}} f(r,\theta,z)\,\Delta V$$

what is ΔV?



$$\Delta V = \Delta A \cdot \Delta z$$

polar area in xy

height

$$= r\,\delta r\,\Delta\theta \cdot \Delta z$$

$$\Rightarrow dV = r\,dr\,d\theta\,dz$$

$$\Rightarrow \lim_{\substack{\text{small} \\ \text{wedges}}} \sum f(r,\theta,z) = \iiint f(r,\theta,z)\,r\,dr\,d\theta\,dz$$

Fubini: still works.

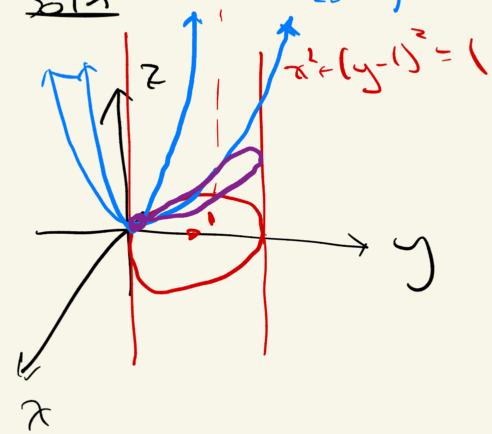
Ex:

Find volume above xy plane inside the cylinder $x^2+(y-1)^2=1$

and below the paraboloid $z=x^2+y^2$

circle centered at $(x,y)=(0,1)$

Soln:



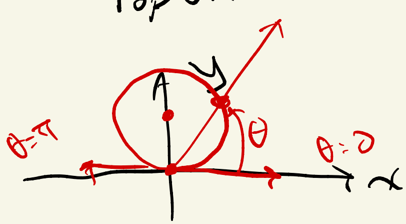
Fix (r,θ). How does z vary?

$z \geq 0$ (above xy plane)

$z \leq x^2+y^2$

$\Rightarrow \boxed{z=r^2}$ is our top

Top view:



shadow in xy:

$x^2+(y-1)^2=1$

$\Rightarrow \underbrace{x^2+y^2}_{r^2}-2y+\cancel{1}=\cancel{1}$

$r^2=2y$

$=2r\sin\theta$

10:57

if $r\neq 0$,

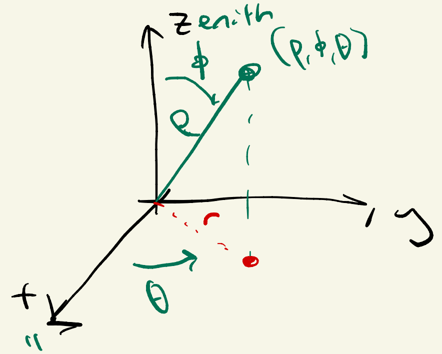
$r=2\sin\theta$

$$\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\sin\theta}\int_{z=0}^{z=r^2} r\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} r\,dz\,dr\,d\theta$$

$$=\frac{3}{2}\pi$$

15.8 Spherical

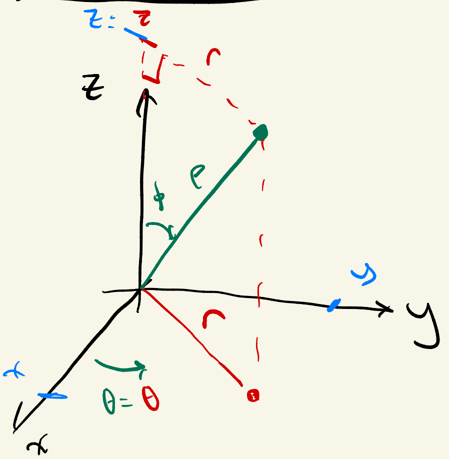
It's like polar... twice!



- $0 \le \rho$ radius
- not beyond π! will see why when we integrate → $0 \le \phi \le \pi$, "inclination" (measured from the "zenith" aka "north pole" usually z axis)
- $0 \le \theta \le 2\pi$ azimuthal angle
- r is same as in polar/cylindrical measured from azimuth (usually from x axis)

warning: physicists use (ρ, θ, ϕ) with
θ inclination and
ϕ azimuth
we will not use this!

Conversions (draw triangles):



Spherical - Cylindrical

$z = \rho\cos\phi$ $\quad \rho^2 = r^2 + z^2$

$r = \rho\sin\phi$ $\quad \phi = \arctan\left(\frac{r}{z}\right)$

$\theta = \theta$

Spherical - Rectangular

$x = r\cos\theta$
$= \rho\sin\phi\cos\theta$

$y = r\sin\theta$
$= \rho\sin\phi\sin\theta$

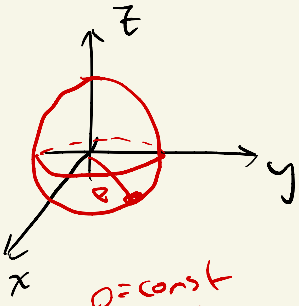
$z = \rho\cos\phi$

$\rho^2 = x^2 + y^2 + z^2$

$\theta = \arctan\left(\frac{y}{x}\right)$

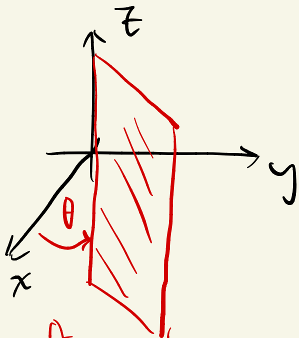
$\phi = \arccos\left(\frac{z}{\rho}\right)$
$= \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

Constant Surfaces?



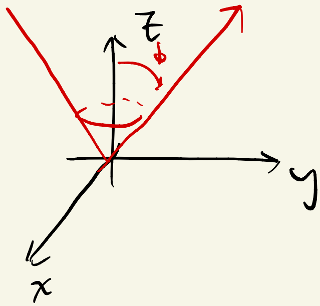
$\rho = \text{const}$

sphere radius ρ



$\theta = \text{const}$

half-plane



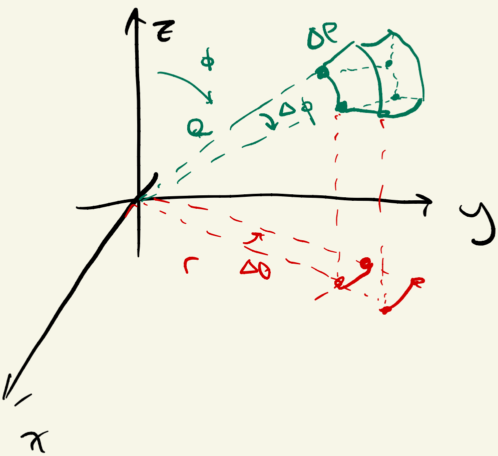
$\phi = \text{const}$

infinite cone

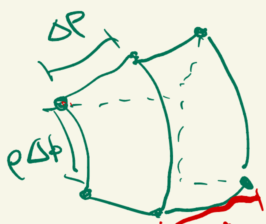
The classic question: what is the volume differential?

Riemann sums

Recall polar



$$\Delta A = \underbrace{r\Delta\theta}\,\Delta r$$



$$r\Delta\theta = \rho\sin\phi\,\Delta\theta$$

$$\Delta V = \underbrace{\Delta\rho}\cdot\underbrace{\rho\Delta\phi}\cdot\underbrace{\rho\sin\phi\,\Delta\theta}$$

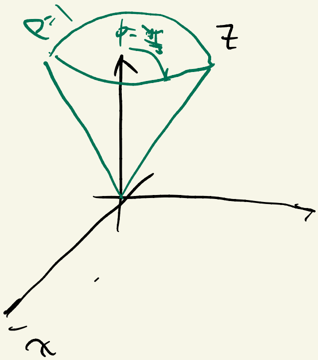
$$= \rho^2\sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta$$

$$\text{So}\quad \sum_{\substack{\text{tiny}\\\text{wedges}}} f(\rho,\phi,\theta)\,\Delta V \approx \iiint_R f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Fubini still works

Ex: Vol of "ice cream cone"

inside $\phi = \frac{\pi}{3}$ and $\rho \leq 1$

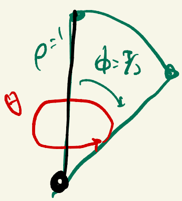


$$V = \iiint_R \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\frac{\pi}{3}} \int_{\rho=0}^{\rho=1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

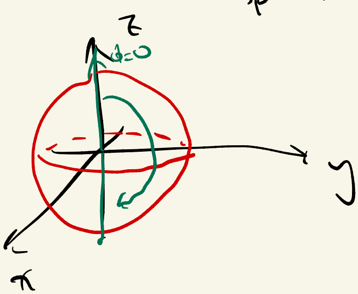
$$= \frac{\pi}{3}$$



warning: Note that $0 \leq \phi \leq \pi$.
For example, $\phi = -\frac{\pi}{2}$ or $\phi = \frac{3\pi}{2}$ is off limits (ruins $\sin\phi$)

Ex: Sphere:

sphere radius R



$$\int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=R} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{4}{3}\pi R^3$$